We shall obtain the equations of PC and $O'P$ using φ as a parameter. Let ON and OM be the perpendiculars from the center O upon PC and $O'P$, respectively. Since $\angle NOO' = \angle NCD = 1/2 \angle DO'P = b/2 + \varphi/4$, $\angle YON = \varphi - b/2 - \varphi/4 = 3\varphi/4 - b/2$; also $\angle YOM = \pi/2 + \varphi/2$ and $\angle NOC = \angle NOO' + a/2 = (a + b)/2 + \varphi/4$. Then $ON = \cos [(a + b)/2 + \varphi/4]$ and $OM = r \sin (\varphi/2)$, where $r = OO'$. Hence the coördinates of P satisfy the equations

$$x \sin \left(\frac{3}{4}\varphi - \frac{b}{2}\right) + y \cos \left(\frac{3}{4}\varphi - \frac{b}{2}\right) = \cos \left(\frac{a+b}{2} + \frac{\varphi}{4}\right),$$

$$x \cos \frac{\varphi}{2} - y \sin \frac{\varphi}{2} = r \sin \frac{\varphi}{2},$$

and by solving these equations we obtain the parametric equations of the locus of P:

$$x = \sin \frac{\varphi}{2} \left[\cos \left(\frac{a+b}{2} + \frac{\varphi}{4}\right) + r \cos \left(\frac{3}{4}\varphi - \frac{b}{2}\right)\right] \Big/ \cos \left(\frac{1}{4}\varphi - \frac{1}{2}b\right),$$

$$y = \left[\cos \frac{\varphi}{2} \cos \left(\frac{a+b}{2} + \frac{\varphi}{4}\right) - r \sin \frac{\varphi}{2} \sin \left(\frac{3}{4}\varphi - \frac{b}{2}\right)\right] \Big/ \cos \left(\frac{1}{4}\varphi - \frac{1}{2}b\right).$$

II. Solution by Otto Dunkel, Washington University.

Let O be the center of the given circle and let us suppose that, when CD lies on the same side of AB as O, the intersection E of DB and CA lies within the circle, and that P is the intersection of the internal bisectors of the angles of the triangle ECD. Indicate by G and H the middle points of the chords CD and AB, respectively, and by O' and S the points in which the bisector EP meets OG and OH. The external bisectors of the angles D and C of the triangle ECD are perpendicular, respectively, to PD and PC, and meet EP in P', the center of the escribed circle in $\angle DEC$. Hence PP' is a diameter of the circle through P, D, P', C, with O' as center. This circle has a constant radius since $\angle DEC$ is constant in magnitude and hence $\angle DPC$ has also a constant value; also OO' is constant in length. Let EP meet CD and AB in T and U, respectively; the construction of the figure shows that the triangles AUE and DTE are similar, and, therefore, that $\angle HUS = \angle GTO'$, and, finally, that $\angle HSU = \angle GO'T$. The triangle OSO' is thus isosceles and S is a fixed point. This determines an easy construction for the curve as follows: With O as center draw a fixed circle of radius OO' passing through S; draw a variable chord SO' and lay off upon it the constant lengths $O'P$ and $O'P'$; then P is the center of the inscribed circle of ECD (in the position mentioned above) and P' is the center of the escribed circle in $\angle DEC$, both of which are points of the locus. The curve is, therefore, the *limaçon of Pascal.*

When C falls on B, P also falls upon B; when D coincides with A, P also coincides with A. When CD or any part of it is on the side of AB opposite to that of O, E and the two points P and P' are all outside of the given circle, the latter two being escribed centers in $\angle CDE$ and $\angle DCE$, respectively. After one revolution of O' and CD, P and P' are interchanged.

The equation of the locus is easily obtained in polar coördinates, taking S as the pole, SO as the axis, $\angle OSP = \theta$, $SP = \rho$. It will be convenient to take the radius OD of the given circle as unity and to denote the lengths of the arcs CD and AB by a and c, respectively, and the acute angle which PC makes with DP by b. Then $b = 1/2 \angle CEB = \pi/2 - (a + c)/4$. In the triangle ODO', $\angle DO'O = b$ and, by the Law of Sines we have $OO' = \sin (b + a/2)/\sin b = \cos (a/4 - c/4)/\cos (a/4 + c/4)$; $O'D = \sin (a/2)/\sin b = \sin (a/2)/\cos (a/4 + c/4)$. Hence

$$\rho = 2\frac{\cos \dfrac{a-c}{4}}{\cos \dfrac{a+c}{4}} \cos \theta - \frac{\sin \dfrac{a}{2}}{\cos \dfrac{a+c}{4}}.$$

Reversing the order of the points C and D causes E to be outside of the given circle in the initial position. In order to obtain the equation for this case we had merely to replace a by $-a$ in the above equation and in the expression for OS. The remaining cases may be treated in a similar manner.

2869 [1921, 36]. Proposed by the late L. G. WELD.

The successive segments of a broken right line are represented by the successive terms of the harmonic progression, 1, 1/2, 1/3, 1/4, *ad infinitum.* Each segment makes with the preceding a

given angle θ. What is the distance and what is the direction of the limiting point (if there be such) from the initial point of the first segment?

SOLUTION BY P. H. GRAHAM, Washington Square College, New York University.

Take the origin of rectangular coördinates as the initial point and let the first segment make an angle θ with the x-axis. Let X and Y be, respectively, the sums of the projections of the segments on the x-axis and on the y-axis; D the distance of the limiting point from the initial point and α the angle which the radius vector to the limiting point makes with the x-axis. Then

$$X = \sum_1^\infty \frac{\cos k\theta}{k}, \qquad Y = \sum_1^\infty \frac{\sin k\theta}{k}, \qquad D = \sqrt{X^2 + Y^2}, \qquad \alpha = \tan^{-1}\frac{Y}{X}. \tag{1}$$

Setting $z = \cos\theta + i\sin\theta$, we have the known development

$$-\log(1 - z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots, \qquad z \neq 1. \tag{2}$$

Hence

$$-\log(1 - \cos\theta - i\sin\theta) = \sum_1^\infty \frac{(\cos\theta + i\sin\theta)^k}{k} = \sum_1^\infty \frac{\cos k\theta + i\sin k\theta}{k}. \tag{3}$$

But $-\log(1 - \cos\theta - i\sin\theta) = -\log(2 - 2\cos\theta)^{1/2} + i\tan^{-1}[\sin\theta/(1 - \cos\theta)]$ and, hence, equating the real and imaginary parts of (3), we have

$$X = -\log(2 - 2\cos\theta)^{1/2} = -\log\left(2\sin\frac{\theta}{2}\right), \qquad Y = +\tan^{-1}\frac{\sin\theta}{1 - \cos\theta} = \frac{\pi - \theta}{2}. \tag{4}$$

Therefore

$$D = \sqrt{\log^2\left(2\sin\frac{\theta}{2}\right) + \left(\frac{\pi - \theta}{2}\right)^2}, \qquad \alpha = \tan^{-1}\frac{\theta - \pi}{2\log\left(2\sin\frac{\theta}{2}\right)}, \qquad 0 < \theta < 2\pi.$$

NOTE ON THE ABOVE SOLUTION BY OTTO DUNKEL, Washington University—The angle θ may be taken so that $0 < \theta < 2\pi$, and the angle of $1 - z$, say ψ, may then be taken so that when $\theta = \pi$, $\psi = 0$, $-\pi/2 < \psi < \pi/2$. Inspection of a figure will show at once that $\psi = (\theta - \pi)/2$ and that the absolute value of $1 - z$ is $2\sin(\theta/2)$, so that

$$\log(1 - z) = \log(2\sin\theta/2) + i(\theta - \pi)/2.$$

The development in (2) is valid for all points on the circle of convergence of the series except for the singular point $z = 1$. The proof of this may be found in Goursat-Hedrick, *A Course in Mathematical Analysis*, vol. 2, part 1, page 19, foot-note, where the convergence of the series is proved, while the argument on pages 20, 21 shows that the series converges to the value on the left in (2). See also pages 38, 39 in the same text for a treatment of $\log(1 + z)$ which gives the above results by a simple substitution.

Also solved by AUGUSTUS BOGARD, R. E. JOHNSON, and ELIJAH SWIFT.

NOTES AND NEWS.

It is to be hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to H. P. MANNING, Brown University, Providence, R. I.

Mr. C. C. PHIPPS, of the University of Montana, has been appointed instructor of mathematics at the University of Minnesota.

Miss MINNA SCHICK, instructor of mathematics at the University of Minne-